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THE PARTITE CONSTRUCTION AND RAMSEY SET SYSTEMS

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This paper deals with Ramsey properties of finite set systems of a given type. We present new and simple proofs of some of the most general results in Ramsey theory for set systems. The proofs rely on a new proof of the Partite Lemma which is combined with an amalgamation technique known as Partite Construction.

Introduction

The following result [18] is one of the most famous and fundamental of combinatorial statements.

Finite Ramsey Theorem. *For every choice of positive integers t, a, b there exists a positive integer c such that* $c \to (b)_t^a$.

Here $c \to (b)_t^a$ is a short hand notation (due to Erdős and Rado) for the following statement:

For every partition of the collection of all a-element subsets of a set X of size c, there exists a b-element subset B of C such that all a-element subsets of B belong to one class of the partition.

This theorem has been generalized many times and several of these generalizations are both profound and difficult to prove. Motivated by general results due to Rado [19] and Graham, Leeb and Rothschild [3], one of the main streams of the research was formed by efforts to prove a very general result which would imply all the known (usually difficult) instances. Thus development culminated with the proof of the *Ramsey theorem for systems,* which we shall state after introducing a few standard notions.

A *type* $\mathbf{\Delta} = (\mathcal{M}_\delta; \delta \in \Delta)$ is an indexed collection of positive integers. Throughout this paper we will fix an index set Δ and a type $\mathbf{\Delta}$.

A system A of type $\mathbf{\Delta}$ is a pair $(X, \mathcal{M})$ where X is a finite linearly ordered set, $\mathcal{M} = (\mathcal{M}_\delta; \sigma \in \Delta)$, and $\mathcal{M}_\delta \subseteq \binom{X}{n_\delta}$. (As customary, here $\binom{X}{k}$ denotes the set of all k-element subsets of X.) We shall suppose that $\mathcal{M}_\delta \cap \mathcal{M}_{\delta'} = \emptyset$ for $\delta \neq \delta'$. Elements of the sets $\mathcal{M}_\delta$ are called *edges* of A.

A is a *subsystem* of $B = (Y, \mathcal{N})$ if X is a subset of Y with the induced order, and $\mathcal{M}_\delta = \mathcal{N}_\delta \cap P(X)$ for every $\delta \in \Delta$.

Two systems $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are *isomorphic* if there is a monotone bijection $f \cdot X \to Y$ taking $\mathcal{M}$ onto $\mathcal{N}$.

A subsystem of B isomorphic to A is called a *copy* of A in B. Denote by $\binom{B}{A}$ the set of all copies of A in B.

A system F is called *irreducible* if every pair of points of F is contained in an edge of F.

The arrow $C \to (B)_t^A$ is defined by analogy with the classical Erdős–Rado case: $C \to (B)_t^A$ if for every partition $\binom{C}{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_t$ there exists $B' \in \binom{C}{B}$ and an i, exists such that $\binom{B'}{A} \subseteq \mathcal{A}_i$. The following result was proved by the authors in [6] and [7].

The Ramsey Theorem for Systems. *Let t be a positive integer and let A, B be systems. There exists a system C such that*

$$C \to (B)_t^A.$$

Moreover, if A, B do not contain an irreducible system F then C may be chosen with the same property.

The original proofs of this result were difficult and complex, see [7, 12]. Let us remark that related results were obtained by Abramson and Harrington and by Prömmel [1, 17]. However, for some special cases (such as partitions of edges of graphs and hypergraphs) several simple proofs were found, see [[13, 14]. These proofs are variations on a common theme – the systematic use of amalgamation of partite systems. This *Amalgamation technique* has been known to authors since 1976 and was effectively used in several papers [10, 13, 14, 15]. This technique did not imply the Ramsey theorem for systems. A breakthrough was achieved in 1980 [16], with a proof of an old conjecture of Erdős related to the Ramsey property of rectangle free graphs. (This seemingly esoteric question is a cornerstone of the area and yields e.g. the existence of complex designs; see the discussion below in Section 3). The purpose of this note is to further extend the methods of [14, 16] and to present a new proof of the Ramsey theorem for systems. (H.J. Prömmel and B. Voigt recently found a different simple proof of this result. Their method is also a variant of the Amalgamation technique.) It appears that our approach is strong and flexible enough to yield virtually all the known Ramsey theorems for special classes of set systems. The paper is divided in three parts: In the first part we derive the Partite Lemma which is the starting point of our amalgamation technique. This part uses one new trick. In the second part we apply the Partite Construction. This part is routine for anyone familiar with the Partite Construction and we closely follow the ideas of [14]. For the convenience of the reader we outline the proof of the basic properties of the Partite Construction.

In the final part we state several strengthenings of the above results which follow from our method.

1. The Partite lemma

An *a-partitie system a* is a pair $((X_i)_{i=1}^a, \mathcal{M})$ where
(i) $X = \bigcup_{i=1}^a X_i$ is an ordered set satisfying $X_1 < X_2 < \cdots < X_a$
(ii) $\mathcal{M} = (\mathcal{M}_\delta; \delta \in \Delta)$, $\mathcal{M}_\delta \subseteq \binom{X}{n_\delta}$
(iii) $|M \cap X_i| \leq 1$ for every $M \in \mathcal{M}_\delta$, $i = 1, \ldots, a$, $\delta \in \Delta$.

Sets X_i are called *parts* of A, elements of M *edges* of A. Property (iii) implies that edges are transversals with respect to the family $X_1 < \cdots < X_a$. Given a subset $Y \subseteq X$ we denote by $\mathrm{tr}(Y)$ the *trace* of Y, i.e. the set $\{i; Y \cap X_i \neq \emptyset\}$.

A is called *transversal* if $|X_i| = 1$ for every $i = 1, \ldots, a$. A is a *subsystem* of $B = ((Y_i)_{i=1}^b, \mathcal{N})$ if there exists a monotone injection $\iota: \{1, \ldots, a\} \to \{1, \ldots., b\}$ such that $X_i \subseteq Y_{\iota(i)}$ for $i = 1, \ldots, a$ and

$$\mathcal{M}_\delta = \mathcal{N}_\delta \cap \binom{X}{n_\delta} \quad \text{for } \sigma \in \Delta.$$

As before $((X_i)_{i=1}^a, \mathcal{M})$ is *isomorphic* to $((Y_i)_{i=1}^a, \mathcal{N})$ if there is an order preserving bijection from $X = \bigcup_i X_i$ to $Y = \bigcup_i Y_i$ sending each part X_i onto the corresponding Y_i and taking $\mathcal{M}$ to $\mathcal{N}$. A subsystem of B isomorphic to A is called a *copy* of A in B; the set of such copies is denoted again by $\binom{B}{A}$. The arrow notation carries over in the obvious way.

The Partite lemma. *Let t be a positive integer and let A and B be a-partite systems. Moreover, let A be transversal. Then there exists an a-partite system C such that*

$$C \to (B)_t^A.$$

Proof. Put $A = ((X_i)_{i=1}^a, \mathcal{M})$, $B = ((Y_i)_{i=1}^a, \mathcal{N})$. As A is transversal we may suppose without loss of generality that $\bigcup_{\delta \in \Delta} \mathcal{M}_\delta$ is the set of all subsets of X (this may be achieved by adding a set of "dummy" edges to $\mathcal{M}$ and $\mathcal{N}$).

Without loss of generality we may also suppose that every vertex $y \in Y$ is contained in a copy of A. This is a general comment (see e.g. [9]): if B^* is the subsystem of B induced by $\binom{B}{A}$ and $C^* \to (B^*)_t^A$ then we can easily construct a system C such that $C \to (B)_t^A$ by enlarging every $B^* \in \binom{C^*}{B^*}$ to a system B. Now take N to be a sufficiently large (indeed very large) number; the actual value of N will be estimate later. Define an a-partite system $C = ((Z_i)_{i=1}^a, \mathcal{O})$, $\mathcal{O} = (\mathcal{O}_\delta; \delta \in \Delta)$ as follows. Set $Z_i = Y_i \times \cdots \times Y_i$ (N times); i.e. each element of Z_i has the form $(x_j; x_j \in Y_i, j = 1, \ldots, N)$. Put $Z = \bigcup_{i=1}^a Z_i$ and for $j = 1, \ldots, N$ define a projection $\pi_j: Z \to Y$ by $\pi_j(x_k; k = 1, \ldots, N) = x_j$. Clearly π_j maps Z_k to Y_k.

We define $\mathcal{O} = (\mathcal{O}_\delta; \delta \in \Delta)$ as follows. First put $\mathcal{N}_\delta = \mathcal{N}'_\delta \cup \mathcal{N}''_\delta$ where $\mathcal{N}'_\delta$ is the set of all edges of $\mathcal{N}$ which belongs to a copy of A in B, $\mathcal{N}''_\delta = \mathcal{N}_\delta - \mathcal{N}'_\delta$ (note that we cannot assume that $\mathcal{N}''_\delta = \emptyset$).

The set $\{(x_j^k; j=1,\ldots,N); k=1,\ldots,n_\delta\}$ belongs to $\mathcal{O}_\delta$ if $\operatorname{tr}(\{x_j^k; k=1,\ldots,n_\delta\}) = \operatorname{tr}(\{x_{j'}^k; k=1,\ldots,n_\delta\})$ for all $j, j' \leq N$ and one of the following possibilities occurs:

(1) $\{x_j^k; k=1,\ldots,n_\delta\} \in \mathcal{N}'_\delta$ for every $j=1,\ldots,N$;

(2) there exists a non-empty set $\omega \subseteq \{1,\ldots,N\}$ and a set $\{x^k; k=1,\ldots,n_\delta\} \in \mathcal{N}''_\delta$ such that $\{x_j^k; k=1,\ldots,n_\delta\} = \{x^k; k=1,\ldots,n_\delta\}$ for all $j \in \omega$ and for all $j \notin \omega$ there is some η with

$$\{x_j^k; k=1,\ldots,n_\delta\} \in \mathcal{N}'_\eta.$$

(Note that, in general, $\eta \neq \delta$, however η is determined by $\operatorname{tr}(\{x_j^k; k=1,\ldots,n_\delta\})$. We shall prove that $C \to (B)_t^A$ provided N is large enough. This follows easily from the following two facts.

Fact 1. $A' \in \binom{C}{A}$ iff $\pi_j(A') \in \binom{B}{A}$ for every $j=1,\ldots,N$.

Proof. Obvious from the definition of $\mathcal{O}$. □

In order to state Fact 2, we introduce some notation.

Put $\binom{B}{A} = \{A_1,\ldots,A_r\}$. Put $R=\{1,\ldots,r\}$. Think of the set R^N endowed with Hales–Jewett (combinatorial) lines. A *line* is a set L of the following form: Fix $\omega \subseteq \{1,\ldots,N\}$ and $\alpha^0 = (\alpha_1^0,\ldots,\alpha_N^0) \in R^N$ and put

$$L = \{(\alpha_1,\ldots,\alpha_N); \quad \alpha_i = \alpha_i^0 \quad \text{for } i \notin \omega, \quad \alpha_i = \alpha_j \quad \text{for } i, j \in \omega.$$

Clearly $|L| = r$. Given $\alpha = (\alpha_1,\ldots,\alpha_N) \in R^N$ denote by $V(\alpha)$ the set of all vertices $\boldsymbol{x}$ of Z which satisfy $\pi_j(\boldsymbol{x}) \in A_{\alpha_j}$. Put $V(L) = \bigcup_{\alpha\in L} V(\alpha)$. By virtue of Fact 1 the set $\binom{C}{A}$ is in 1–1 correspondence with the set R^N.

Fact 2. Let L be a line of R^N. Then $V(L)$ induces a copy of B in C.

Proof. Check the definition of $V(L)$. □

Now we invoke the classical Hales–Jewett theorem [5] and choose N sufficiently large such that for every partition of R^N into t classes one of the classes contains a monochromatic line. This implies $C \to (B)_t^A$. Indeed, let $\binom{C}{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_t$ be a partition. By Fact 1 this induces a partition $R^N = \mathcal{A}'_1 \cup \cdots \cup \mathcal{A}'_t$ by $\alpha \in \mathcal{A}'_i$ iff $V(\alpha)$ induces a copy belonging to $\mathcal{A}_i$. By the Hales–Jewett theorem there exists a monochromatic line l in R^N. This in turn, using Fact 2, $B' \in \binom{C}{B}$ such that $\binom{B'}{A}$ is monochromatic. □

2. The Partite construction

In this part we prove the Ramsey theorem for systems by means of the Partite Construction. We follow closely the construction given in [14, 13].

Let k, A, B be fixed. Consider A as a transversal a-partite system and B as a transversal b-partite system. Explicitly

$$B = (\{y_1, \ldots, y_b\}, \mathcal{M}).$$

Put $p = r(a, t, b) = \min\{n; n \to (b)^a_t\}$ (the classical Ramsey number). Put $q = \binom{p}{a}$, $\binom{\{1,\ldots,p\}}{a} = \{M^1, \ldots, M^q\}$. We shall construct "pictures" $P^0, \ldots, P^k, \ldots, P^q$ by induction on k. Picture P^q will be the desired system C. Let $P^0 = ((X^0_i)^p_{i=1}, \mathcal{O}^0)$ be a p-partite system where for each choice of b parts $X^0_{i_1}, \ldots, X^0_{i_b}$ the subsystem of P^0 induced by them contains a copy of B. Such a "picture" P^0 may be formed as a disjoint union of copies of B. Suppose pictures $P^k = ((X^k_i)^p_{i=1}, \mathcal{O}^k)$, $k < q$, be given. Consider M^{k+1} and the a-partite system D^{k+1} induced in P^k by parts X^k_i where y_i belongs to M^{k+1}.

By the Partite lemma there exists an a-partite system E^{k+1} such that

$$E^{k+1} \to (D^{k+1})^A_t.$$

Extend each copy of D^{k+1} in E^{k+1} to a copy of P^k in such a way that the distinct copies of P^k intersect in vertices of E^{k+1} only. In this amalgamation the parts of distinct copies of P^k are preserved. Denote the resulting amalgamation by $\binom{E^{k+1}}{D^{k+1}} * P^k$. (If a more explicit definition is needed, see [15]). Put $P^{k+1} = \binom{E^{k+1}}{D^{k+1}} * P^k$. Finally, put $C = P^q$. We claim that C has the desired properties.

Claim 1. Every irreducible subsystem in C is a subsystem of B.

Proof. Induction on k. This being trivial for $k = 0$, in the inductive step the amalgamation does not create any new irreducible system. □

Claim 2. $C \to (B)^A_t$.

Proof. We apply backward induction on $k = q, q-1, \ldots, 1$. In the inductive step $(k+1 \to k)$ we apply the Partite lemma and find a copy of P^k such that all copies of A with trace A^k are monochromatic. this leaves us, for $k = 0$, with a copy P' of P^0 in C where the color of a copy of A in P depends only on its trace. However, such a copy of P^0 contains a monochromatic copy of B by the construction of P^0 and the fact that $p \to (b)^a_t$. □

3. Applications

The Partite construction yields, in the spirit of [9] and [11], several results stronger than the Ramsey theorem for systems. We list some of them.

A. *Hom-connected graphs*

First we give an auxiliary definition. Let $B = (X, \mathcal{M})$ be a set system. A set $Y \subset X$ is called a *cut* of B if there is a partition of $X - Y$ into two disjoint sets

Y_1, Y_2 such that no pair $\{y_1, y_2\}$, $y_1 \in Y_1$, $y_2 \in Y_2$, is covered by an edge of B. We shall also consider the cut Y as a subsystem of B determined by Y.

A system B is called hom *A-connected* if no cut Y of B has a homomorphism into A. Here a homomorphism is an edge-preserving mapping. (The notion of a Hom K_k-connected graph coincides with the notion of chromatically k-connected graph, see [15].)

It is also convenient to recall the following notion from [9]: Given a (possibly infinite) set $\mathscr{F}$ of systems, denote by Forb($\mathscr{F}$) the set of all those system A which do not contain any system $F \in \mathscr{F}$ as a weak subsystem of B. (A is a *weak subsystem* of B if every vertex (edge, respectively) of A is a vertex (edge, respectively). Now we have the following result.

Theorem 3.1. *Let $\mathscr{F}$ be a set of* hom *A-connected systems. Then for every positive t and every $B \in$* Forb($\mathscr{F}$) *there exists $C \in$* Forb($\mathscr{F}$) *such that $C \to (B)_t^A$.*

Proof. First fix $P = (T, \mathscr{P})$ such that $P \to (B)_t^A$. Put $T = \{1, \ldots, p\}$, $\binom{P}{B} = \{B_1, \ldots, B_q\}$. Define picture $P = ((X_i^0)_{i=1}^p, \mathcal{M}^0)$ by:

$$X_i = \{i\} \times \{1, \ldots, q\}$$

and

$$\{(i_j, m_j); j = 1, \ldots, n_\delta\} \in \mathcal{M}_\delta^0 \text{ iff}$$

$$k = m_j = m_{j'} \text{ for } j, j' = 1, \ldots, q,$$

$$\{i_j; j = 1, \ldots, n_\delta\}$$

is an edge of B and belongs to $\mathcal{N}_\delta$. (Thus P^0 is a disjoint union of copies $B_1, \ldots, B_q$ with traces induced by $\binom{P}{B}$).

Clearly $P^0 \in$ Forb($\mathscr{F}$) and the amalgamation does not create any new hom A connected subsystem of P^{k+1}. (Note that every a-partite system $D^1, \ldots, D^q$ may be homomorphically mapped into A.) $\square$

The Partite Construction is very convenient for constructing sparse Ramsey graphs. This is not surprising as one of the byproducts of the partite construction is a new easy construction of highly chromatic graphs without short cycles [10].

There are various ways of defining sparseness and we list them in the order of increasing difficulty.

B. *Sparse Ramsey theorems – Ramsey families*

We say that $\mathscr{B} \subseteq \binom{C}{B}$ is a *Ramsey family* if for every partition

$$\binom{C}{A} = \mathscr{A}_1 \cup \cdots \cup \mathscr{A}_t$$

there exists $B' \in \mathscr{B}$ such that $\binom{B'}{A} \subseteq \mathscr{A}_i$ for some i. We denote this by $C \xrightarrow{\mathscr{B}} (B)_t^A$.

We associate with $\mathscr{B}$ a uniform hypergraph $H^A_{\mathscr{B}} = (X, E)$ where $X = \binom{C}{A}$ and

$$E = \left\{ \binom{B}{A};\quad B \in \mathscr{B} \right\}.$$

We have the following result for sparse Ramsey families.

Theorem 3.2. *For A, B and positive integers t, l there exist C and a system $\mathscr{B} \subseteq \binom{C}{B}$ such that*

(1) $C \xrightarrow{\mathscr{B}} (B)^A_t$

(2) The hypergraph $H^A_{\mathscr{B}}$ has no cycles of length $\leqslant l$.

Proof. We proceed by induction on l and construct $\mathscr{B}$ by means of the Partite Construction (see Section 2). Put $\mathscr{B}^0 = \binom{P^0}{A}$. In the inductive step we assume that there exists a system $\mathscr{D}^{k+1} \subseteq \binom{E^{k+1}}{D^{k+1}}$ such that $H^A_{\mathscr{D}^{k+1}}$ has no cycles of length $\leqslant l - 1$.

We form the picture $P^{k+1} = \mathscr{D}^{k+1} * P^k$. Assuming that in P^k we have a system $\mathscr{B}^k$ then in P^{k+1} we may define a system $\mathscr{B}^{k+1}$ as $\mathscr{D}^{k+1} * \mathscr{B}^k$ consisting of copies of B which have no cycles of length $\leqslant l$ (it appears that all copies in $\mathscr{B}^0$ and thus $\mathscr{B}^{k+1}$ are transversal). See [15] where this argument is covered in details. $\square$

C. *Sparse Ramsey theorems – cycles in copies*

We prove the following result concerning sparse copies.

Theorem 3.3. *Let Be be a* Hom *A-connected system. Let t, l be positive integers. Then there exists a set system C with the following properties*:

(1) $C \to (B)^A_t$

(2) *The hypergraph $H^A_{\binom{C}{B}}$ has no cycles of length $\leqslant l$.*

Proof. We apply the Partite Construction. We start with picture P^0 introduced in the above proof of Theorem 3.1 We proceed by induction on k. In the inductive step we use Theorem 3.2 to obtain a Ramsey system $\mathscr{D}^{k+1} \subseteq \binom{E^{k+1}}{D^{k+1}}$ without cycles of length $\leqslant l$. Putting $P^{k+1} = \mathscr{D}^{k+1} * P^k$ one can check, as in the proof of Theorem 3.2, that the hypergraph $H^A_{\binom{P^{k+1}}{B}}$ has no cycles of length $\leqslant l$. $\square$

D. *Linearity*

We say that a system B is *A-linear* if any two copies of A in B intersect in at most one vertex. Typical examples of linear systems are Steiner systems. In [16] we proved a Ramsey theorem for Steiner systems. More generally, we have the following.

Theorem 3.4. *Let A be a system t positive integer. Then for every A-linear B there exists A-linear C such that*

$$C \to (B)^A_t.$$

Proof. Check the construction in the Proof of the Partite lemma. Use the fact that the Hales–Jewett lines form a linear system. This implies that two copies of B in C intersect either in a copy of A or in a subset of one part of C. Use this fact in Partite Construction: as all copies of B in all pictures are transversal, prove linearity of B by induction on $P^0, P^1, \ldots, P^k$. □

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